Energy and time resolution of the $180^\circ$ hemispherical electrostatic analyser

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Abstract The energy resolution and transit time spread of the $180^\circ$ hemispherical electrostatic analyser for non-relativistic electrons or ions have been computed. The results have been expressed in terms of the dimensions of the analyser and the parameters describing the input particle beam for a large range of possible configurations. The predictions for a particular electron analyser have been tested experimentally and good agreement has been obtained.

1 Introduction
Electrostatic energy analysers have become important tools in many branches of collision physics and in photoelectron spectrometry. Furthermore, several recent applications have involved the use of such analysers in coincidence experiments where both the variation in the transit times of the electrons through the analyser and the energy resolution are of importance (e.g. Amaldi et al. 1969, Ehrhardt et al. 1969, Imhof and Read 1971, King and Adams 1974). The energy resolutions of such analysers have previously been calculated in various approximate ways (e.g. Kuyatt and Simpson 1967, Roy and Carette 1971, Read et al. 1974), but no data on transit time spreads exist. This has been a serious deficiency since, as will become apparent, the design criteria for good energy resolution and good time resolution are in conflict and a suitable compromise must be made.

We have calculated these properties numerically for the $180^\circ$ hemispherical electrostatic analyser (Purcell 1938, Simpson 1964). In these calculations the fringing fields in the entrance and exit planes have been neglected. These have been discussed by Purcell (1938) and more recently by Wollnik (1965, 1967), where methods of designing the entrance geometry to minimize their effects are given. We have also neglected space charge effects as they are unlikely to be important in analysers (as distinct from monochromators) where the current densities are usually very low. Relativistic effects have also been neglected.

2 The analyser
The analyser considered here is shown schematically in figure 1. It consists of two concentric hemispherical electrodes of radii...
Energy and time resolution of electrostatic analyser

A cross section through a 180° electrostatic hemispherical analyser. The hemispherical electrodes have radii \( r_1 \) and \( r_2 \) with a common centre at 0 and are at potentials \( V_1 \) and \( V_2 \). The entrance and exit apertures are circular, of diameter \( d \), centred a distance \( r_0 = \frac{1}{2}(r_1 + r_2) \) from 0. The other parameters are defined in the text.

\[
k = 2ZeE_0r_0.
\]  

(1)

If all the potentials are measured with respect to the zero of the kinetic energy of these particles, then the voltages on the inner and outer electrodes are

\[
V_1 = E_0(2r_0/r_1 - 1) \quad \text{and} \quad V_2 = E_0(2r_0/r_2 - 1).
\]

(2)

In traversing the field boundary at \( \theta = 0 \), both the kinetic energy, \( ZeE_0 \), and the angle of incidence, \( \alpha \), are changed to \( ZeE' \) and \( \alpha' \) where (Purcell 1938)

\[
E' = E + \delta E \\
\delta E = 2E_0(r_0/r_1 - 1)
\]

(3)

and

\[
\sin \alpha' = (1 + \delta E/E)^{-1/2} \sin \alpha.
\]

A similar effect occurs in the exit plane at \( \theta = \pi \), but this does not affect the present calculations. The equation of motion inside the deflector can be written in the form (see Goldstein 1951, for example)

\[
1/r = (mk/\lambda)(1 + \epsilon \cos (\theta - \phi))
\]

(4)

where \( m \) is the mass of the particle, \( \lambda \) is the angular momentum, given by

\[
I^2 = 2ZeE_0[r_0(\cos^2 \alpha + 2E_0(r_0/r_1 - 1))]
\]

(5)

and \( \epsilon \) is the eccentricity, given by

\[
e^2 = 1 + 2ZeE_0(2E_0)/mk^2.
\]

(6)

The angle, \( \phi \), is determined from the initial conditions at \( \theta = 0 \). From equation (4) it can easily be deduced that the magnitudes of the radius vectors in the entrance and exit planes, \( r_1 \) and \( r_r \), are related by

\[
1/r_1 = 2mk/\lambda^2 - 1/r_1.
\]

(7)

The transit times, \( T \), of the particles through the analyser are given by

\[
T = \frac{2mA}{mk^2} \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 + \epsilon \cos (\theta - \phi)^2}}
\]

(8)

where \( A \) is the area enclosed by the orbit within the analyser and is evaluated using equation (4). This integral is elementary, but we used a binomial expansion of the integrand, retaining terms up to the fourth power in the eccentricity. Thus

\[
T \approx \frac{15\pi e^4}{8} \left( \pi - 2e \sin \phi + \frac{3\pi e^2}{2} - 8e^2 \sin \phi + \frac{15\pi e^4}{8} - \ldots \right).
\]

The largest inaccuracy of this approximation was estimated to be less than 0.01 % for the range of eccentricities encountered in the present calculations.

3 Method of computation

In designing an analyser, the physical parameters to be determined are the dimensions \( r_0 \) and \( d \), the mean kinetic energy \( ZeE_0 \) and the range of angles \( \alpha \) the beam of particles subtends at the entrance aperture. In the computations we represent the beam by conical pencils of rays of semi-angle \( \alpha_{max} \), with their axes parallel to the \( z \) axis and their apices in the plane of the entrance aperture. This corresponds to the usual mode of operation of such analysers, where the incoming beam is focused on to the entrance aperture (real or virtual), with the pupil at infinity.

The pencil angle, \( \alpha_{max} \), is more conveniently expressed in terms of the dimensionless ratio \( \alpha_{max}/\alpha_0 \), where \( \alpha_0 \) is defined by

\[
\alpha_0 = (d/4r_0)^{1/2}.
\]

(9)

The usual design criteria (Kuyatt and Simpson 1967) require the pencil angle to be limited to \( \alpha_{max}/\alpha_0 \leq 1 \). Our computations cover the range \( \alpha_{max}/\alpha_0 \leq 2 \), with the further limitation \( \alpha_{max} \leq 0.24 \) rad. This should include most practical applications.

In determining whether a particle is transmitted by the analyser, it is only necessary to determine whether the magnitude of the radius vector in the exit plane, \( r_r \), is within the values defined by the exit aperture. The circular input aperture is therefore divided into a mesh of at least 200 points. At each point the input energy is varied over a sufficiently wide range to
include all possible energies that can be transmitted. At each energy and mesh point, \( \alpha \) is varied from \(-\alpha_{\text{max}}\) to \(\alpha_{\text{max}}\) and \( r_f \) is calculated using equation (7). A contribution to the transmitted current is then added, if \( r_f \) falls within the range defined by the circular exit aperture. The size of this contribution is proportional to the solid angle subtended by the element of the pencil in the entrance plane. The transit time spread is determined by computing the transit time, using equation (8), of each trajectory which contributes to the transmitted current.

4 Energy resolution

The energy profiles of transmitted particles for an analyser with \( d/r_0 = 0.05 \) and a range of pencil angles are shown in figure 2.

![Image](url)

**Figure 2** Energy profiles of transmitted particles for an analyser with \( d/r_0 = 0.05 \) and a range of pencil angles: A, \( \alpha_{\text{max}}/\alpha_0 = 0 \); B, \( \alpha_{\text{max}}/\alpha_0 = 1 \); C, \( \alpha_{\text{max}}/\alpha_0 = 2 \)

Other values of \( d/r_0 \) yield the same essential features. Several of these features can be understood by referring to equations (5) and (7). The maximum and minimum energies transmitted are, to first order in \( d/r_0 \) and second order in \( \alpha_{\text{max}} \),

\[
E_{\text{min}} \cong E_0 (1 - d/2r_0 + \ldots) \\
E_{\text{max}} \cong E_0 (1 + d/2r_0 + 3\alpha_{\text{max}}^2 + \ldots). 
\]

In figure 2 it is seen that the minimum transmitted energy is not a function of \( \alpha_{\text{max}} \), whereas the mean transmitted energy is. This follows from equations (10). For small \( \alpha_{\text{max}} \) the base width is approximately equal to \( E_0 d/r_0 \) and, assuming the energy profile to be symmetrical and triangular, the full width at half height, \( \Delta E(0.5) \) is given by \( E_0 d/2r_0 \), as derived by Kuyatt and Simpson (1967).

We have parameterized \( \Delta E(0.5)/E_0 \) for the range of values \( d/r_0 \leq 1 \) and \( \alpha_{\text{max}}/\alpha_0 \leq 2 \), with the further restriction \( \alpha_{\text{max}} < 0.24 \) rad. The form of the parameterization is that suggested by equations (10) but the numerical constants have been found by a least squares fit to the data obtained from the computed energy profiles. Our parameterization is

\[
\Delta E(0.5)/E_0 = 0.43 d/r_0 + 0.25 \alpha_{\text{max}}^2. 
\]

The accuracy of this formula is better than 1%. Read *et al.* (1974) derived a comparable, though less accurate result by a different method.

In figure 2 it can be seen that there is a long, low intensity tail on the high energy side of the profiles. We therefore do not give base widths but, in figure 3, plot the width at 10% of the peak height, \( \Delta E(0.1)/\Delta E(0.5) \), and at 1% of the peak height, \( \Delta E(0.01)/\Delta E(0.5) \). Only data for \( d/r_0 = 0.05 \) are presented, as the variation with this parameter is less than 2% over the range investigated. Large values of \( d/r_0 \) give slightly larger normalized widths at 10% and 1%.

5 Time resolution

The transit time profiles for an analyser with \( d/r_0 = 0.05 \) and a range of pencil angles are shown in figure 4. The widths of these curves depend more strongly on the pencil angle than the

![Image](url)

**Figure 3** Variation of the widths at 10% (\( \Delta E(0.1)/\Delta E(0.5) \), curve A) and at 1% (\( \Delta E(0.01)/\Delta E(0.5) \), curve B) of the peak height with pencil angle

![Image](url)

**Figure 4** Time profiles of transmitted particles for an analyser with \( d/r_0 = 0.05 \) and a range of pencil angles: A, \( \alpha_{\text{max}}/\alpha_0 = 0 \); B, \( \alpha_{\text{max}}/\alpha_0 = 1 \); C, \( \alpha_{\text{max}}/\alpha_0 = 2 \)
corresponding widths of the energy profiles. Curve A corresponds to a parallel input beam and its base width can be calculated approximately from the difference in transit times, $\Delta T$, of two circular orbits with radii, $r_0 + d/2$ and $r_0 - d/2$. This is given, to first order in $d/r_0$ by

$$\Delta T/T_0 \approx 3d/2r_0$$  \hspace{1cm} (12)$$

where

$$T_0 = \pi r_0 (m/2ZeE_0)^{1/2}. \hspace{1cm} (13)$$

$T_0$ is the transit time of a particle of kinetic energy $ZeE_0$ travelling in a circular orbit of radius $r_0$.

An approximate expression relating the transit time spread to the pencil angle can be obtained by calculating the difference in transit times of two trajectories, with angles of incidence of $\pm \alpha_{\text{max}}$ and passing through the centres of the entrance and exit apertures. In this case

$$\Delta T/T_0 \approx \left(8/\pi\right)\alpha_{\text{max}}$$  \hspace{1cm} (14)$$
to first order in $\alpha_{\text{max}}$. The first two terms of equation (8) were used in deriving this formula. Equation (14) is more useful in practice than equation (12), because the transit time spread depends only weakly upon the ratio $d/r_0$. Indeed, a very convenient approximate formula for the full width at half height, $\Delta T(0.5)$ of the computed time profiles is

$$\Delta T(0.5)/T_0 \approx 2.1 \alpha_{\text{max}}.$$  \hspace{1cm} (15)$$

This expression gives results correct to 15% for the range $d/r_0 \leq 0.1$, $0.5 \leq \alpha_{\text{max}}/\alpha_0 \leq 2$ and $\alpha_{\text{max}} \leq 0.24$ rad. A more accurate formula, valid for the whole range of pencil angles is

$$T(0.5)/T_0 \approx 0.60d/r_0 + 2.23(1 - 2.79d/r_0)\alpha_{\text{max}}.$$  \hspace{1cm} (16)$$
The constants in the expressions (15) and (16) were found by a least squares procedure, using the computed transit time spread curves. Equation (16) gives values which are within 5% of the computed values.

Again we do not consider it useful to give base widths, so the widths at 10% ($\Delta T(0.1)/\Delta T(0.5)$) and at 1% ($\Delta T(0.01)/\Delta T(0.5)$) of the peak height are presented in figure 5.

6 Experimental investigation

The validity of equations (11) and (16) was tested experimentally with an apparatus normally employed to measure atomic lifetimes using the electron–photon delayed coincidence technique (King and Adams 1974). Here both good time and energy resolution are required. Two hemispherical electrostatic deflectors having a mean radius, $r_0$, of 25.4 mm are used in the apparatus and these are typically operated at an analysing energy of 5 eV. One deflector, the monochromator, is used to obtain a monoenergetic beam of electrons from a dispenser cathode while the other, the energy analyser, is used to analyse inelastically scattered electrons. It is this energy analyser that needs to have both good energy and time resolution. In these studies the current in the analyser was $\sim 1$ nA, three orders of magnitude lower than the critical current where space charge effects become important. The relationships (equations (11) and (16)) were investigated by finding the energy and time resolutions as a function of the analysing energy.

From equations (11), (13) and (16) it may be seen that if both $\alpha_{\text{max}}$ and $d/r_0$ are kept constant, as was done in the present studies, then

$$\Delta T(0.5)/T_0 \approx \left(8/\pi\right)\alpha_{\text{max}}.$$  \hspace{1cm} (14)$$

and

$$\Delta E(0.5) \propto E_0$$  \hspace{1cm} (17a)$$

$$\Delta T(0.5) \propto E_0^{-1/2}.$$  \hspace{1cm} (17b)$$

The observed energy profile is the convolution of the transmission profiles of the monochromator and analyser. If we approximate these profiles to gaussian curves their full widths at half height add in quadrature to give the observed energy resolution $\Delta E_{\text{obs}}$. Equation (17a) can be modified to give

$$\Delta E_{\text{obs}}^2 = \alpha E_0^3 + \Delta E_M^2$$  \hspace{1cm} (18)$$

where $\Delta E_M$ is the monochromator resolution and $\alpha$ is a constant. A range of analysing energies from 4 eV to 18 eV was used and the results are shown in figure 6. The intercept on the ordinate gives the square of the energy resolution of the monochromator. The data are well fitted by a straight line, confirming the functional relationship of equation (18).

![Figure 5](image1.png)  
**Figure 5** Variation of the widths at 10% ($\Delta T(0.1)/\Delta T(0.5)$, full curves) and 1% ($\Delta T(0.01)/\Delta T(0.5)$, broken curves) of the peak height with pencil angle for three values of $d/r_0$: A, $d/r_0 = 0.01$; B, $d/r_0 = 0.05$; C, $d/r_0 = 0.1$

![Figure 6](image2.png)  
**Figure 6** Variation of the observed energy resolution $\Delta E_{\text{obs}}$ of a hemispherical analyser with mean kinetic energy $E_0$.
The observed time resolution as a function of the reciprocal of the square root of the analysing energy is shown in figure 7. The results were obtained by measuring the lifetime of the $6^1P$ state in mercury which is small compared to our time resolution. (Lurio (1965) measured a value of $1.31 \pm 0.08 \text{ ns}$.) Our instrumental profile was found by analysing the time spectrum using Gale’s equation (Gale 1962) which is a gaussian instrumental profile convoluted with an exponential decay. The measured values of the full width at half height were corrected for a constant residual time resolution of 3 ns due to the detectors. We again assumed this added in quadrature. These data are also well fitted by a straight line, confirming the functional relationship of equation (17b).

7 Conclusions

We have computed the energy and time resolution of the 180° hemispherical electrostatic analyser over a wide range of parameters. The characteristics have been presented in graphical form and in terms of convenient design equations. We have also verified the validity of these expressions with one particular electron energy analyser.

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